The Existence of Energy Minimizers for Knots and Links

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Abstract

In this paper, I will introduce the mathematical ideas of knots and links, define a function on each that we call the energy of a knot or link, and prove several lemmas and theorems that relate the knot and link energies to basic concepts in knot theory. The main results of this paper will be the proofs of two theorems on the existence of energy minimizers for both knots and links. Finally, I will compare the depth and structure of these two major proofs in order to better understand the nature of the results.

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Contents

1 Introduction

Here I will briefly discuss the nature and structure of my paper. This introduction will be divided into three sections: first, the basic definitions related to knots and links; second, an introduction to the specifics of knot and link energies and the main results of my paper; third, the analytic tools that will help prove the results. For the first section, the statements of definitions are modified from those in Lickorish [11]. The statements of the second section come from Freedman, He, and Wang [6] and He [8], and those of the third section are from both Freedman, He, and Wang [6] and Bartle [1].

1.1 Knots and Links

Naturally, to start this paper, we must first define precisely what we mean by a knot and a link. Mathematical knots are slightly different than a knot that we would tie with a piece of string, in that mathematical knots have no ends; they are embedded circles in $\mathbb{R}^3$. It is as if we took a piece of string, tied it into a standard knot, and glued the ends together. A link is then just how it sounds: two knots "linked" together. For our purposes, all knots and links will be smooth; that is, they have continuous derivatives of all orders. The precise definitions follow.

Definition 1.1 *A knot is a smooth simple closed curve in* $\mathbb{R}^3$*; i.e. it is a smooth embedding* $\gamma : \mathbb{S}^1 \to \mathbb{R}^3$.

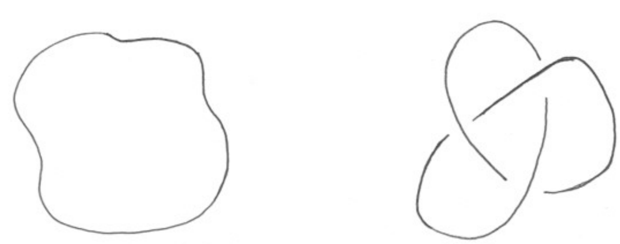

Figure 1: Unknot (left) and trefoil (right)

Definition 1.2 *A two-component link (hereafter, a link) is the union of two disjoint knots. We write a link in terms of its components, i.e. the link* (γ_1, γ_2) *consists of the disjoint knots* γ_1 *and* γ_2*. We say* (γ_1, γ_2) *is a non-trivial link if there is no smoothly embedded 2-sphere* S *in* $\mathbb{R}^3$ *such that each component of* $\mathbb{R}^3 \setminus S$ *contains exactly one component of the link,* γ_1 *or* γ_2*.*

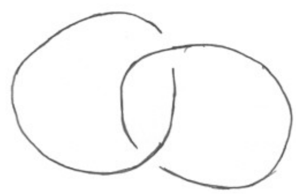

Figure 2: Hopf link

An important idea in knot theory is the idea of two knots being "ambiently isotopic" to each other. Intuitively, one knot is ambiently isotopic to another knot if it can be continuously deformed in $\mathbb{R}^3$ to the other knot; i.e. there is a transformation of $\mathbb{R}^3$ that sends one knot to the other. More precisely, we can give the following definition:

Definition 1.3 *Two knots* γ_1 *and* γ_2 *are ambiently isotopic if there exists an orientation-preserving diffeomorphism* $h : \mathbb{R}^3 \to \mathbb{R}^3$ *such that* $h(\gamma_1) = \gamma_2$*.*

The main motivation for the notion of ambient isotopy is in order to define a knot class.

Definition 1.4 *Two knots* γ *and* η *are in the same knot class* K *if they are ambiently isotopic.*

1.2 Knot and Link Energies

Now that we have some of the basic ideas behind knot theory, we can move on to the specifics related to knot and link energies, which are the focus of this paper. The energy of a knot was first introduced in a 1991 paper by O'Hara entitled "Energy of a knot" [15] as a function that measures the complexity of a particular embedding of a knot: the energy is small for "nice" embeddings, such as the standard embedding of the circle into $\mathbb{R}^3$, and increases as the complexity of the embedding increases. We will go into more detail about the energy later in this paper. Let us now define the energy of a knot.

Definition 1.5 *The energy E of the knot γ is defined as*

$$E(\gamma) = \int_{\mathbb{S}^1}\int_{\mathbb{S}^1} \left\{ \frac{1}{|\gamma(v)-\gamma(u)|^2} - \frac{1}{D(\gamma(v),\gamma(u))^2} \right\} |\gamma\prime(u)||\gamma\prime(v)| du dv, \quad (1)$$

where $D(\gamma(u),\gamma(v))$ denotes the arc length distance between $\gamma(u)$ and $\gamma(v)$ along the knot.

If we parametrize the knot by arc length, then if l is the length of the curve we can rewrite the energy equation as

$$E(\gamma) = \int_{-l/2}^{l/2}\int_{x-l/2}^{x+l/2} \left\{ \frac{1}{|\gamma(y)-\gamma(x)|^2} - \frac{1}{|y-x|^2} \right\} dy dx. \quad (2)$$

This formula will prove easier to use when calculating the energy of a knot, since we will usually parametrize the knot by arc length.

It is proved in O'Hara [15] that these integrals converge and that the energy is a well-defined continuous function with respect to the $\mathcal{C}^2$-topology. We can see from the definition that the local contribution to the energy will be small when the Euclidean distance between two points on the knot is close to the arc-length distance between the same two points or when the lengths of the tangent vectors to the knot are small. Likewise, the local energy contribution will be large when two points on the knot are separated by a great distance in arc length but are close together in $\mathbb{R}^3$, or when the lengths of the tangent vectors are large. In figure 3 below, we see these different local contributions of the energy. In the arc on the left, the Euclidean distance between the two points a and b is very close to the arc-length distance between a and b on the knot, so the energy is small. In the arc on the right, the points a and b are much closer in Euclidean distance than they are in arc-length distance, so the energy is larger.

We will now calculate an example in order to become familiar with these types of calculations.

Example: *Energy of a circle*

Figure 3: Arc with small energy (left) and with large energy (right)

Let γ be an arc-length parametrization of the circle with radius 1; so the length of the circle is 2π. Then if $|y-x| \leq \pi$,

$$|\gamma(y) - \gamma(x)| = 2\sin(|y-x|/2). \tag{3}$$

So, by equation 2 we have

$$E(\gamma) = \int_{-\pi}^{\pi} \int_{x-\pi}^{x+\pi} \left\{ \frac{1}{[2\sin(|y-x|/2)]^2} - \frac{1}{(y-x)^2} \right\} dydx. \tag{4}$$

Then we can break up the second integral into two parts, and we have

$$\begin{aligned} E(\gamma) &= \int_{-\pi}^{\pi} \int_{x-\pi}^{x} \left\{ \frac{1}{[2\sin(|y-x|/2)]^2} - \frac{1}{(y-x)^2} \right\} dydx \\ &+ \int_{-\pi}^{\pi} \int_{x}^{x+\pi} \left\{ \frac{1}{[2\sin(|y-x|/2)]^2} - \frac{1}{(y-x)^2} \right\} dydx. \end{aligned}$$

Using a u-substitution with $u = \frac{y-x}{2}$ we get

$$\begin{aligned} E(\gamma) &= 2\pi \int_{-\pi/2}^{0} \left\{ \frac{1}{4\sin^2(-u)} - \frac{1}{4u^2} \right\} 2du \\ &+ 2\pi \int_{0}^{\pi/2} \left\{ \frac{1}{4\sin^2(u)} - \frac{1}{4u^2} \right\} 2du \end{aligned}$$

$$
\begin{aligned}
&= \pi \int_{-\pi/2}^{0} \left\{ \frac{1}{\sin^2(-u)} - \frac{1}{u^2} \right\} du + \pi \int_{0}^{\pi/2} \left\{ \frac{1}{\sin^2(u)} - \frac{1}{u^2} \right\} du \\
&= \pi \left(-\cot(-u) + \frac{1}{u} \right) |_{-\pi/2}^{0} + \pi \left(-\cot(u) + \frac{1}{u} \right) |_{0}^{\pi/2},
\end{aligned}
$$

and since $\cot u = 1/u - (1/3)u + \dots$ by a Taylor series expansion about 0, we have $\cot u - \frac{1}{u} \to 0$ as $u \to 0$, hence

$$
\begin{aligned}
E(\gamma) &= \pi \left(\frac{2}{\pi} \right) + \pi \left(\frac{2}{\pi} \right) \\
&= 4.
\end{aligned}
$$

So the energy of a circle is 4.

The definition of a link energy is similar to that of a knot, and is introduced in a paper by He entitled "On the minimizers of the Mobius cross energy of links" [8]. We give the definition below.

Definition 1.6 *The energy E of the link (γ_1, γ_2) is defined as*

$$
E(\gamma_1, \gamma_2) = \int_{\mathbb{S}^1} \int_{\mathbb{S}^1} \frac{|\gamma_1'(u)||\gamma_2'(v)|}{|\gamma_1(u) - \gamma_2(v)|^2} du dv. \tag{5}
$$

We can see from this definition that the link energy will be large when the components are close together in $\mathbb{R}^3$ and small when they are far apart. We also see that this definition does not contain the arc length term in the knot energy, because the link energy is not concerned with the energy contributed locally by each component; we only consider the inter-component energy.

Now that we have defined the energies for knots and links, we can give the statements of the main results of this paper. First, we have a theorem of He [8] for links that proves the existence of a non-trivial link with minimal energy.

Theorem 1.7 *There exists a non-trivial link (η_1, η_2) such that $E(\eta_1, \eta_2) \leq E(\gamma_1, \gamma_2)$ for any non-trivial link (γ_1, γ_2) with $|(\gamma_i)'(u)| \neq 0$ for $i = 1, 2$.*

Second, we have a similar theorem of Freedman, He, and Wang [6] for knots that proves the existence of a minimal energy knot for every knot class.

Theorem 1.8 *Given a knot class K, there exists a knot γ_K in K such that $E(\gamma_K) \leq E(\gamma)$ for any other knot γ in K.*

These theorems will be proved in sections 2 and 3.2, respectively.

1.3 Geometric and Analytic Tools

In order to actually compute knot and link energies and to prove any sort of interesting results about these energies, we must have at our disposal a few tools of geometry and analysis.

1.3.1 Mobius Transformations

First, we define the notion of a Mobius transformation, which will be useful in actually calculating the energies of knots and links. For our purposes, Mobius transformations will be transformations that act on $\mathbb{R}^3 \cup \{\infty\}$. This definition is adapted from that in Lang [10].

Definition 1.9 *A Mobius transformation of* $\mathbb{R}^3 \cup \{\infty\}$ *is a function defined by the composition of functions of three types:*
(1) Translations along a vector v: $T_v(x) = x + v$
(2) Multiplications by a constant c: $M_c(x) = cx$
(3) Inversions about the origin: $I(x) = \frac{x}{|x|^2}$

The reason that Mobius transformations are so useful in calculating energies is that the energy is in fact invariant under Mobius transformation. Both the knot and the link energies are invariant, but we will only prove the theorem for the link energy. The statement and proof of this theorem is expanded from a statement in He [8].

Theorem 1.10 *Let* $(\gamma_1, \gamma_2), \gamma_i : \mathbb{S}^1 \to \mathbb{R}^3$ *be a link and let* T *be a Mobius transformation of* $\mathbb{R}^3 \cup \{\infty\}$. *If* $(T(\gamma_1), T(\gamma_2)) \subseteq \mathbb{R}^3$, *then* $E(T(\gamma_1), T(\gamma_2)) = E(\gamma_1, \gamma_2)$.

Proof: We must prove the following formula:

$$\frac{|T'_x||T'_y|}{|T(x) - T(y)|^2} = \frac{1}{|x - y|^2}, \tag{6}$$

where $|T'_x| = \lim_{t \to 0} \frac{|T(x+th) - T(x)|}{|th|}$ for any $h \in \mathbb{R}^3 \setminus \{0\}$. Then we will have that

$$\begin{aligned} E(T(\gamma_1), T(\gamma_2)) &= \int\int \frac{|\partial/\partial u(T(\gamma_1'(u))||\partial/\partial v(T(\gamma_2'(v))|}{|T(\gamma_1(u)) - T(\gamma_2(v))|^2} \\ &= \int\int \frac{|T'(\gamma_1'(u))(\gamma_1'(u))||T'(\gamma_1'(v))(\gamma_2'(v))|}{|\gamma_1(u) - \gamma_2(v)|^2} \\ &= \int\int \frac{|\gamma_1'(u)||\gamma_2'(v)|}{|\gamma_1(u) - \gamma_2(v)|^2}. \end{aligned}$$

We will prove this equation for each type of Mobius transformation: (1) translations, (2) dilations, and (3) inversions. Since all Mobius transformations are compositions of these types of functions, the theorem will then be proved.

(1) Let $T(x) = x + v$. We see that

$$\begin{aligned}|T'_x| &= \lim_{t\to 0} \frac{|T(x+th) - T(x)|}{|th|} \\ &= \lim_{t\to 0} \frac{|(x+v+th) - (x+v)|}{|th|} \\ &= \lim_{t\to 0} \frac{|th|}{|th|} \\ &= 1,\end{aligned}$$

and the same holds for $|T'_y|$, so we have

$$\begin{aligned}\frac{|T'_x||T'_y|}{|T(x) - T(y)|^2} &= \frac{1}{|(x+v) - (y+v)|^2} \\ &= \frac{1}{|x-y|^2},\end{aligned}$$

(2) Let $T(x) = cx$. Then we have

$$\begin{aligned}|T'_x| &= \lim_{t\to 0} \frac{|T(x+th) - T(x)|}{|th|} \\ &= \lim_{t\to 0} \frac{|(cx+cth) - cx|}{|th|} \\ &= \lim_{t\to 0} \frac{|cth|}{|th|} \\ &= |c|,\end{aligned}$$

and again the same holds for $|T'_y|$. So, we have

$$\begin{aligned}\frac{|T'_x||T'_y|}{|T(x) - T(y)|^2} &= \frac{|c||c|}{|cx - cy|^2} \\ &= \frac{|c|^2}{|c|^2|x-y|^2} \\ &= \frac{1}{|x-y|^2},\end{aligned}$$

(3) Let $T(x) = \frac{x}{|x|^2}$. By a calculation in Beardon [2], we have that

$$|T(x) - T(y)| = \frac{|y-x|}{|x||y|}, \tag{7}$$

and we can calculate

$$\begin{aligned} |T'_x| &= \lim_{t\to 0} \frac{|T(x+th) - T(x)|}{|th|} \\ &= \lim_{t\to 0} \frac{|x - (x+th)|}{|x+th||x|} \cdot \frac{1}{|th|} \\ &= \lim_{t\to 0} \frac{1}{|x+th||x|} \\ &= \frac{1}{|x|^2}. \end{aligned}$$

So, we have

$$\begin{aligned} \frac{|T'_x||T'_y|}{|T(x) - T(y)|^2} &= \frac{1}{|x|^2} \cdot \frac{1}{|y|^2} \cdot \frac{|x|^2|y|^2}{|y-x|^2} \\ &= \frac{1}{|y-x|^2}, \end{aligned}$$

and the proof is complete. □

1.3.2 Analytic Tools

In order to prove some of the results related to knot energies, it is important for us to have a definition of a certain characterization of functions. This characterization is somewhat similar to that of uniform continuity, but is a stronger condition since it forces the distance between function values to be bounded by a constant multiple of the distance between the domain values. We define what it means for a function to be L-Lipschitz or L-bi-Lipschitz. The statements and sketch of a proof in this section are modified from those in Freedman, He, and Wang [6].

Definition 1.11 *Let X and Y be sets with metrics d_X and d_Y, respectively. A map $f : X \to Y$ is called L-Lipschitz if $d_Y(f(u), f(v)) \leq L \cdot d_X(u,v)$ for all $u, v \in X$. It is called L-bi-Lipschitz if f is L-Lipschitz and its inverse $f^{-1} : f(X) \to X$ exists and is also L-Lipschitz.*

A lemma and its corollary from Freedman, He, and Wang [6] (Lemma 1.2 and Corollary 1.3) that relate the L-bi-Lipschitz property of functions to the energy of a knot are critical to the main results of this paper. We will give the statements below and a sketch of the proof of the lemma (the corollary follows immediately). For more details, the interested reader can consult Freedman, He, and Wang [6].

Lemma 1.12 *Let γ be a knot parametrized by arc length. If $E(\gamma)$ is finite, then the function $\gamma : \mathbb{S}^1 \to \mathbb{R}^3$ is L-bi-Lipschitz with the bi-Lipschitz constant $L = L(\gamma)$ depending only on $E(\gamma)$. Additionally, $L(\gamma)$ converges to 1 as $E(\gamma)$ goes to 0.*

Sketch: Since γ is parametrized by arc length, we know that the map γ is L-Lipschitz with L= 1. So, we must show that it is in fact L-bi-Lipschitz for some L (not necessarily 1). Then we will let the bi-Lipschitz constant be the larger of 1 and L. Since γ is a knot, hence an embedding, the inverse must exist, so we must show that there exists an L dependent only on $E(\gamma)$ such that for all $u, v \in \mathbb{S}^1$,

$$|u - v| \leq L \cdot |\gamma(u) - \gamma(v)|. \tag{8}$$

Again, since γ is parametrized by arc length, $|u - v| = D(\gamma(u), \gamma(v))$, so we want to show that

$$D(\gamma(u), \gamma(v)) \leq L \cdot |\gamma(u) - \gamma(v)|. \tag{9}$$

Intuitively, we would expect this lemma to hold based on the definition of the knot energy. By this definition, for the first part of the lemma we see that the finiteness of the energy will allow us to bound the arc-length distance between points on the knot by some multiple of the Euclidean distance, and therefore obtain a bi-Lipschitz constant dependent on the energy. For the second part, we know that the local contribution to the energy will be small when the Euclidean distance between two points on the knot is close to the arc-length distance between the same two points, so as the energy gets small, $D(\gamma(u), \gamma(v))$ and $|\gamma(v) - \gamma(u)|$ will become closer together, and therefore the bi-Lipschitz constant will go to 1 as the energy goes to 0.

Corollary 1.13 *Let $\gamma : \mathbb{S}^1 \to \mathbb{R}^3$ be a knot with finite energy. Then for any $\epsilon > 0$, there exists $\delta > 0$ such that any subarc γ_δ of length δ is a $(1 + \epsilon)$-bi-Lipschitz embedding under the arc-length parametrization.*

As an aside, we note that an L-Lipschitz or L-bi-Lipschitz function is necessarily uniformly continuous.

1.3.3 Two Famous Theorems

We will now state two well-known theorems that will be very useful in proving our main results. Since these theorems have a completely different feel from the rest of this paper, we will not prove them here. First, we state Fatou's Lemma, a lemma from the Lebesgue theory of integration. The reader can consult, for example, Capinski and Kopp [5] for a proof.

Theorem 1.14 (Fatou's Lemma) *If $\{f_n\}$ is a sequence of non-negative integrable functions, then*

$$\liminf_{n\to\infty} \int f_n dm \geq \int \left(\liminf_{n\to\infty} f_n \right) dm. \tag{10}$$

Next, we state the Arzela-Ascoli Theorem. Consult, for example, Bartle [1] for a proof.

Theorem 1.15 (Arzela-Ascoli Theorem) *Let $K \subset \mathbb{R}^p$ be compact and let $\mathcal{F}$ be a collection of continuous fuctions $f : K \to \mathbb{R}^n$. Then every sequence from $\mathcal{F}$ has a uniformly convergent subsequence if and only if the family $\mathcal{F}$ is bounded and uniformly equicontinuous on K.*

2 Minimization of Link Energies

Since the main result for links requires less background and is easier to digest, we will start with link energies instead of knot energies. First, so we can get a better handle on the definition of the link energy, we will calculate two examples. We see that Theorem 1.10 comes into play here in order to make the calculations easier.

Example: *Energy of Hopf link*

The Hopf link is the link of two round circles, in this case the circles of radius 1 lying in the yz-plane and xy-plane centered at $(0, 1, 0)$ and $(0, 0, 0)$, respectively. Since energy is invariant under Mobius transformations, we can parametrize the first component of the Hopf link by the extended line through the origin along the z-axis, which can be mapped to a circle of radius 1 lying in the yz-plane centered at $(0, 1, 0)$. So, we parametrize as follows:

$$\begin{aligned} \gamma_1(u) &= (0, 0, u) \\ \gamma_2(v) &= (\cos v, \sin v, 0). \end{aligned}$$

Then we have

$$
\begin{aligned}
E(\gamma_1, \gamma_2) &= \int\int \frac{|\gamma_1'(u)||\gamma_2'(v)|}{|\gamma_1(u) - \gamma_2(v)|^2} du dv \\
&= \int_{-\infty}^{\infty}\int_{0}^{2\pi} \frac{1}{|(0,0,u) - (\cos v, \sin v, 0)|^2} dv du \\
&= \int_{-\infty}^{\infty}\int_{0}^{2\pi} \frac{1}{1+u^2} dv du \\
&= 2\pi^2 \approx 19.7392
\end{aligned}
$$

Example: *Energy of off-center Hopf link*

Now we want to see what happens when we change one of the link components slightly. We again consider the link of two round circles of radius 1, with the first lying in the yz-plane centered at $(0, 1, 0)$, and the second lying in the xy-plane centered at $(0, -0.5, 0)$. So instead of the first component passing through the center of the second component, we shift the second component slightly along the x axis. We have the following parametrizations:

$$
\begin{aligned}
\gamma_1(u) &= (0, 0, u) \\
\gamma_2(v) &= (\cos v + 0.5, \sin v, 0).
\end{aligned}
$$

Then we have

$$
\begin{aligned}
E(\gamma_1, \gamma_2) &= \int\int \frac{|\gamma_1'(u)||\gamma_2'(v)|}{|\gamma_1(u) - \gamma_2(v)|^2} du dv \\
&= \int_{-\infty}^{\infty}\int_{0}^{2\pi} \frac{1}{|(0,0,u) - (\cos v + 0.5, \sin v, 0)|^2} dv du \\
&= \int_{-\infty}^{\infty}\int_{0}^{2\pi} \frac{1}{\cos v + 1.25 + u^2} dv du \\
&\approx 21.1838
\end{aligned}
$$

These results confirm our intuition about the link energy; in the second example, the two loops are pulled apart, so the local energy contribution around where the linking occurs is higher, making the energy greater.

Now that we have the necessary background information on links and link energies along with two worked examples under our belts, we can prove the main theorem on the existence of an energy-minimizing link. For convenience's sake, we restate the theorem here. This proof is modified and expanded from that given in He [8].

Theorem 2.1 *There exists a non-trivial link* (η_1, η_2) *such that* $E(\eta_1, \eta_2) \leq E(\gamma_1, \gamma_2)$ *for any non-trivial link* (γ_1, γ_2) *with* $|(\gamma_i)'(u)| \neq 0$ *for* $i = 1, 2$.

Proof: We will first lay out the structure of the proof, giving the four major steps that lead to the final result. As a notational aside, we denote by $d(\gamma)$ the Euclidean diameter of the knot γ, where we define the Euclidean diameter of a knot γ as $max\left\{|\gamma(u) - \gamma(v)|\,|u, v \in \mathbb{S}^1\right\}$. First, given a sequence of links whose energy converges to the infimum, we show that the energy is uniformly bounded. Second, we will show that we can assume that ∞ is on the first element of each link in the sequence, 0 is on the other element of each link, and the Euclidean diameter of the second element of each link is 1. Third, we will show that the length of the second element of each link is uniformly bounded and that the length of the first element of each link is bounded within any closed ball about the origin. Fourth, using these bounds on the lengths of the link elements, we show that the sequence of links converges to a link with the same properties as above.

(1) Let $\{(\gamma_1{}^n, \gamma_2{}^n)\}$ be a sequence of links with energy approaching the infimum such that $|(\gamma_i{}^n)'(u)| \neq 0$ for all $u \in \mathbb{S}^1$. The sequence $\{E(\gamma_1{}^n, \gamma_2{}^n)\}$ is a decreasing sequence bounded below by 0, so we see that $\lim_{n\to\infty} E(\gamma_1{}^n, \gamma_2{}^n)$ exists, and therefore there exists a uniform bound E on $E(\gamma_1{}^n, \gamma_2{}^n)$, i.e. $E \leq E(\gamma_1{}^n, \gamma_2{}^n)$ for all n.

(2) By means of Mobius transformations, we want to show that $\infty \in \gamma_1{}^n$, $0 \in \gamma_2{}^n$, and $d(\gamma_2{}^n) = 1$. Let $x \in \gamma_1{}^n$ be the point we will send to ∞, and $y \in \gamma_2{}^n$ the point that we will send to 0. To send x to ∞, we must first send x to 0 using a translation and then send it to ∞ using an inversion about 0. So, let T_{-x} be the translation that sends x to 0, and let I be the inversion that sends 0 to ∞. Then we have

$$(I \circ T_{-x})(x) = \infty. \tag{11}$$

Since y has now been sent to $(I \circ T_{-x})(y)$, in order to send y to 0 we must use the translation $T_{-I\circ T_{-x}(y)}$, which sends $(I \circ T_{-x})(y)$ to 0 and keeps ∞ at ∞, i.e.

$$\left(T_{-I\circ T_{-x}} \circ I \circ T_{-x}\right)(y) = 0. \tag{12}$$

To ensure that $d(\gamma_2{}^n) = 1$, we use the multiplication by the inverse of the diameter of $\gamma_2{}^n$, after all of the above transformations have been applied. Let

$$\Phi = T_{-I \circ T_{-x}} \circ I \circ T_{-x}. \tag{13}$$

So, we use the multiplication $M_{\frac{1}{d(\Phi(\gamma_2{}^n))}}$, which keeps ∞ at ∞ and 0 at 0. Now, let

$$\Psi = M_{\frac{1}{d(\Phi(\gamma_2{}^n))}} \circ \Phi, \tag{14}$$

and we have finally that

$$\begin{aligned} \Psi(x) &= \infty \\ \Psi(y) &= 0 \\ d\left(\Psi(\gamma_2{}^n)\right) &= 1. \end{aligned}$$

(3) Now we want to bound the length of each $\gamma_2{}^n$. Consider the balls B_r of radius $r \geq 1$ about the origin. Since $d(\gamma_2{}^n) = 1, \gamma_2{}^n \subset B_1$. We know that $\gamma_1{}^n$ must intersect B_1, because otherwise B_1 would be a sphere separating $\gamma_1{}^n$ and $\gamma_2{}^n$, and then $(\gamma_1{}^n, \gamma_2{}^n)$ would not be linked. Now for each B_r we have that for all u with $\gamma_1{}^n(u) \in B_r$ and for all v,

$$|\gamma_1{}^n(u) - \gamma_2{}^n(v)|^2 \leq (2r)^2 = 4r^2. \tag{15}$$

Then since by part (1) above E is a bound on $E(\gamma_1{}^n, \gamma_2{}^n)$, and by definition if we restrict $\gamma_1{}^n$ to the ball B_r we have

$$\ell(\gamma_1{}^n|_{B_r}) = \int |\left(\gamma_1{}^n|_{B_r}\right)'(t)|dt, \tag{16}$$

we have:

$$\begin{aligned} E &\geq E(\gamma_1{}^n, \gamma_2{}^n) \\ &\geq \int\limits_{v \in \mathbb{S}^1} \int\limits_{u|\gamma_1{}^n(u) \in B_r} \frac{|\left(\gamma_1{}^n\right)'(u)||\left(\gamma_2{}^n\right)'(v)|}{|\gamma_1{}^n(u) - \gamma_2{}^n(v)|^2} dudv \\ &\geq \int\int \frac{|\left(\gamma_1{}^n\right)'(u)||\left(\gamma_2{}^n\right)'(v)|}{4r^2} dudv \\ &= \frac{1}{4r^2} \cdot \ell(\gamma_1{}^n|_{B_r}) \cdot \ell(\gamma_2{}^n), \end{aligned}$$

and we have the following inequality relating $\ell(\gamma_1{}^n|_{B_r})$ and $\ell(\gamma_2{}^n)$:

$$\ell(\gamma_2{}^n) \leq \frac{E \cdot 4r^2}{\ell(\gamma_1{}^n|_{B_r})} \tag{17}$$

Since $\infty \in {\gamma_1}^n$, we have an arc of ${\gamma_1}^n$ connecting ∞ to every ball B_r, and so for any $r > 1$, we see that $\ell({\gamma_1}^n|_{B_r}) \geq 2(r-1)$, since ${\gamma_1}^n$ must pass through B_r, into B_1, and back out through B_r, which implies that

$$\ell({\gamma_2}^n) \leq \frac{E \cdot 2r}{2(r-1)} = \frac{Er}{r-1}, \tag{18}$$

so we have a bound for $\ell({\gamma_2}^n)$.

Now we want to obtain a bound for $\ell({\gamma_1}^n|_{B_r})$ for all r. Manipulating equation 17 we can conclude that

$$\ell({\gamma_1}^n|_{B_r}) \leq \frac{E \cdot 4r^2}{\ell({\gamma_2}^n)}, \tag{19}$$

and since $d({\gamma_2}^n) = 1$, we know that $\ell({\gamma_2}^n) \geq 2$ and so we can conclude that

$$\ell({\gamma_1}^n|_{B_r}) \leq \frac{E \cdot 4r^2}{2} = 2Er^2, \tag{20}$$

so we also have a bound on $\ell({\gamma_1}^n|_{B_r})$ for all r.

(4) We want to show that the sequence of links satisfy the conditions for the Arzela-Ascoli Theorem (Theorem 1.15), so that we can obtain a subsequence $\{({\gamma_1}^{n_i}, {\gamma_2}^{n_i})\}$ that converges locally uniformly to the pair (η_1, η_2), with $\infty \in \eta_1, 0 \in \eta_2$, and $d(\eta_2) = 1$. So, we must show that the family of functions that define the links is locally bounded and locally uniformly equicontinuous. From above, we know that ${\gamma_2}^n$ is uniformly bounded because its diameter is equal to 1. We also have that given any neighborhood, ${\gamma_1}^n$ is bounded within that neighborhood, since ${\gamma_1}^n|_{B_r}$ is a mapping into the bounded set B_r. Now we will show that the family of links is locally uniformly equicontinuous. We will consider only ${\gamma_1}^n$, and the same argument follows for ${\gamma_2}^n$. We want to show that given any $\epsilon > 0$, there exists a $\delta > 0$ such that for any n, if $|x - y| < \delta$ then $|{\gamma_1}^n(x) - {\gamma_1}^n(y)| < \epsilon$. Equivalently, if we can find some M such that $|{\gamma_1}^n(x) - {\gamma_1}^n(y)| < M \cdot |x - y|$, then given $\epsilon > 0$ we can let $\delta = \frac{\epsilon}{M}$. First, we will find a reparametrization such that the derivative is bounded above. Since the length of ${\gamma_1}^n$ is locally bounded above by some $L > 0$ (when ${\gamma_1}^n$ is inside the ball of radius r, we have $L = 2Er^2$), we can reparametrize each ${\gamma_1}^n$ on the interval $[0, L]$, using a constant-speed parametrization $\tilde{\gamma_1}^n : [0, L] \to \mathbb{R}^3$. While the specific parametrization is obviously dependent on n, the key is that for all n we can parametrize over the same interval $[0, L]$. Since we have this common domain of parametrization, we can find a constant-speed reparametrization

such that $\left|(\tilde{\gamma_1}^n)'(s)\right| \leq 1$ for all $s \in [0, L]$ and for all n. This parametrization is given by the solution to the differential equation

$$\left|(\gamma_1{}^n)'(f(s))\right| \cdot f'(s) = \frac{\ell(\gamma_1)}{L}, \tag{21}$$

which always has a solution because $\left|(\gamma_1{}^n)'(s)\right| \neq 0$. So, we have a parametrization in which the length of the derivative is bounded above by 1.

By the Fundamental Theorem of Calculus, we have the following equation:

$$\gamma_1{}^n(s_1) - \gamma_1{}^n(s_0) = \int_{s_0}^{s_1} (\gamma_1{}^n)'(s)ds, \tag{22}$$

and since $\tilde{\gamma_1}^n$ is just a reparametrization of $\gamma_1{}^n$, we have

$$\gamma_1{}^n(s_1) - \gamma_1{}^n(s_0) = \tilde{\gamma_1}^n(t_1) - \tilde{\gamma_1}^n(t_0) = \int_{t_0}^{t_1} (\tilde{\gamma_1}^n)'(t)dt. \tag{23}$$

Taking the length of both sides, we see that

$$\begin{aligned} |\gamma_1{}^n(s_1) - \gamma_1{}^n(s_0)| &= \left|\int_{t_0}^{t_1} (\tilde{\gamma_1}^n)'(t)dt\right| \\ &\leq \int_{t_0}^{t_1} \left|(\tilde{\gamma_1}^n)'(t)\right| dt \\ &\leq \int_{t_0}^{t_1} 1 \cdot dt \\ &\leq |t_1 - t_0|, \end{aligned}$$

and so we can let $M = 1$ and therefore the family $\{\gamma_1{}^n\}$ is locally uniformly equicontinuous. The same argument holds for $\gamma_2{}^n$, so we have that the sequence of links is locally uniformly equicontinuous, and we satisfy the conditions for the Arzela-Ascoli Theorem. Therefore, we can obtain a subsequence $\{(\gamma_1{}^{n_i}, \gamma_2{}^{n_i})\}$ that converges locally uniformly to the pair (η_1, η_2).

Now we must show that this pair is in fact nontrivially linked; that is, the linking did not disappear when taking the limit. Suppose that (η_1, η_2) is unlinked. Then there exists some embedded 2-sphere S such that γ_1 is completely contained in one component of $\mathbb{R}^3 \setminus S$ and γ_2 is completely contained in the other component. Since S is a compact set, γ_1 is a closed set, and $S \cap \gamma_1 = \emptyset$, the distance from S to γ_1 is greater than 0; that is, there is some $r_1 > 0$ such that $|x - y| > r_1$ whenever $x \in S$ and $y \in \gamma_1$. Similarly, there exists some r_2 with the same property for γ_2. So, we can construct a regular tubular neighborhood $\mathcal{M}_1$ about γ_1 that is completely contained in one component of $\mathbb{R}^3 \setminus S$ and a regular tubular neighborhood $\mathcal{M}_2$ about γ_2 that is completely contained in the other component. Since the subsequence $\{({\gamma_1}^{n_i}, {\gamma_2}^{n_i})\}$ converges to (η_1, η_2), there is some $N > 0$ such that if $n_i \geq N$ then ${\gamma_1}^{n_i}$ is contained in $\mathcal{M}_1$ and ${\gamma_2}^{n_i}$ is contained in $\mathcal{M}_2$. Therefore the embedded sphere S separates the link $({\gamma_1}^{n_i}, {\gamma_2}^{n_i})$, hence it is unlinked, which is a contradiction. Hence (η_1, η_2) is nontrivially linked.

Now, consider equation 5. Denote by $H_{(\gamma_1,\gamma_2)}(u, v)$ the integrand of this equation. Since the subsequence $\{({\gamma_1}^{n_i}, {\gamma_2}^{n_i})\}$ converges locally uniformly to (η_1, η_2), we have

$$H_{(\eta_1,\eta_2)}(u, v) = \lim_{n_i \to \infty} H_{({\gamma_1}^{n_i},{\gamma_2}^{n_i})}(u, v). \tag{24}$$

From equation 5 we can see that $H_{({\gamma_1}^{n_i},{\gamma_2}^{n_i})}(u, v)$ is a non-negative function for all n_i, so we can apply Fatou's Lemma (Theorem 1.14) to conclude that

$$\begin{aligned}
E(\eta_1, \eta_2) &= \int\int H_{(\eta_1,\eta_2)}(u, v)dudv \\
&= \int\int \lim_{n_i \to \infty} H_{({\gamma_1}^{n_i},{\gamma_2}^{n_i})}(u, v)dudv \\
&= \int\int \liminf_{n_i \to \infty} H_{({\gamma_1}^{n_i},{\gamma_2}^{n_i})}(u, v)dudv \\
&\leq \liminf_{n_i \to \infty} \int\int H_{({\gamma_1}^{n_i},{\gamma_2}^{n_i})}(u, v)dudv \\
&= \liminf_{n_i \to \infty} E({\gamma_1}^{n_i}, {\gamma_2}^{n_i}),
\end{aligned}$$

hence (η_1, η_2) is an energy minimizer, and the theorem is proved. □

3 Knots

3.1 Deeper Results in Knots and Knot Energies

Before delving into the main result of this paper, we will give a bit more background on the energy of knots and their properties. We have defined the the knot energy in section 1.2, and here we will define tameness and the crossing number of a knot, and relate these ideas to the energy of a knot.

The results that we obtain from this section will be used later to prove the existence of an energy minimizing knot for each knot class. The definitions and theorems of this section come from Freedman, He, and Wang [6].

One of the first definitions that we need in order to prove the preliminary theorems is the idea of a tame knot. Intuitively, we know what a tame knot should look like: it should look like a knot that we could conceivably tie with an actual piece of string. A knot that is not tame (or "wild") is one that "wiggles around" very tightly in a neighborhood (see figure 4 below). This knot consists of a sequence of knots tied into an unknot, with the individual knots getting smaller and smaller.

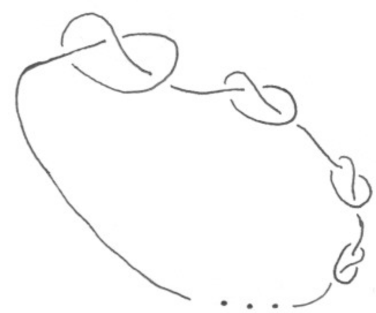

Figure 4: Wild knot

What we want to be able to do with a tame knot is to surround the knot with a small tube that follows every curve of the knot, without the thickened knot self-intersecting (see figure 5 below). That is, if we continue the string analogy, we should be able to enclose the string in a plastic tube without the tube intersecting itself. In figure 4 we can see that no matter how thin we make the tubular neighborhood, at some point the knots will be so small that the tube must self-intersect, so the knot is wild.

The precise mathematical definition of a tame knot follows.

Definition 3.1 *A knot* $\gamma : \mathbb{S}^1 \to \mathbb{R}^3$ *is tame if there exists an extension of the embedding* $\mathbb{S}^1 \times \{0\} \to \mathbb{R}^3$ *to an embedding* $\mathbb{S}^1 \times \mathbb{R}^2 \to \mathbb{R}^3$.

Another characterization of knots is similarly intuitive (although the actual definition is not). Since a knot may (and in the interesting cases,

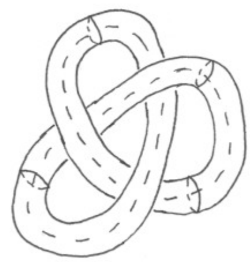

Figure 5: Knot with a tubular neighborhood

generally will) cross behind or in front of itself in $\mathbb{R}^3$, and therefore cross itself in a projection onto $\mathbb{R}^2$, we have the notion of the crossing number of a knot.

Definition 3.2 *Let $\gamma : \mathbb{S}^1 \to \mathbb{R}^3$ be a knot. The average crossing number $c(\gamma)$ of γ is*

$$c(\gamma) = c(\gamma, \gamma) = \frac{1}{4\pi} \int\int \frac{|(\gamma'(x), \gamma'(y), \gamma(y) - \gamma(x)|}{|\gamma(y) - \gamma(x)|^3} dxdy. \tag{25}$$

This definition can be related more directly to the intuitive idea of self-crossings in projections onto $\mathbb{R}^2$ by the following lemma. We denote by $n(\gamma; \theta)$ the number of self-crossings of the knot γ when projected in the θ-direction. By θ-direction we mean some $\theta = (\theta_1, \theta_2) \in \mathbb{S}^2$, and we call a projection in the θ-direction, $\pi_\theta : \mathbb{R}^3 \to \mathbb{R}^2$, a θ-projection when it projects $\mathbb{R}^3$ onto the plane orthogonal to the vector θ. The proof of this lemma is modified from a proof in Freedman, He, and Wang [6].

Lemma 3.3 *Let $\gamma : \mathbb{S}^1 \to \mathbb{R}^3$ be a knot. Then the average crossing number of γ is the number of self-crossings of its θ-projections averaged over $\theta \in \mathbb{S}^2$:*

$$c(\gamma) = \frac{1}{4\pi} \int\int_{\theta \in \mathbb{S}^2} n(\gamma; \theta) dS, \tag{26}$$

where dS denotes the area form on $\mathbb{S}^2$.

Proof: We define a function $F : \mathbb{S}^1 \times \mathbb{S}^1 \setminus \textit{diagonal} \to \mathbb{S}^2$ as follows:

$$F(x, y) = \frac{\gamma(x) - \gamma(y)}{|\gamma(x) - \gamma(y)|}. \tag{27}$$

After a messy calculation involving the Jacobian of this function, we have that

$$|det(dF)| = \frac{|(\gamma'(x), \gamma'(y), \gamma(y) - \gamma(x))|}{|\gamma(y) - \gamma(x)|^3},$$

which is exactly the integrand in the definition of average crossing number. Therefore, substituting into equation 25 yields:

$$c(\gamma) = \frac{1}{4\pi} \int \int |det(dF)| dx dy. \tag{28}$$

Since $|det(dF)|$ is the change of coordinates factor for the function F, we know that $c(\gamma)$ is the area of the (unsigned) images of F divided by 4π. Now, F maps vectors in $\mathbb{R}^3$ that go from one point on the knot to another to unit vectors in the same direction, and the number of vectors that are mapped in a given direction is exactly the number of times γ crosses itself in the projection in that direction. That is, if there is a vector from one point on the knot to another, then if we look at a planar projection in that direction, these two points will be mapped to the same point, producing a self-crossing of the knot in this projection. In figure 6 below, we see in the left-hand figure that the vectors from a to b and from a' to b' are in the same direction, and when we project the knot in this direction as in the right-hand figure, there are two self-crossings.

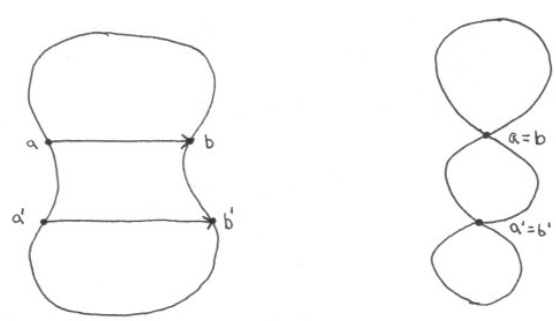

Figure 6: Two projections of the unknot

So, the cardinality of $F^{-1}(\theta)$ is the number of crossings in the direction θ, which is $n(\gamma; \theta)$. This means that the area of the unsigned images of F

equals the integral of $n(\gamma;\theta)$, so we have

$$\int\int |det(dF)|dxdy = \int\int_{\theta\in\mathbb{S}^2} n(\gamma;\theta)dS, \tag{29}$$

which implies

$$c(\gamma) = \frac{1}{4\pi}\int\int |det(dF)|dxdy = \frac{1}{4\pi}\int\int_{\theta\in\mathbb{S}^2} n(\gamma;\theta)dS, \tag{30}$$

and therefore we have equation 26 and the lemma is proved. □

Since we are concerned with knot energies, it is natural to look for a way to incorporate the average crossing number of a knot into an energy calculation. If we think of both the energy and the crossing number as measures of the complexity of a knot, we have the idea that knots of a certain energy can only have a certain number of crossings. A knot with a large crossing number will most likely have a high energy, while a knot with low energy cannot have a large number of crossings. The following lemma gives a bound on the crossing number of a knot in terms of its energy. We give here a sketch of the proof given by Freedman, He, and Wang [6].

Lemma 3.4 *Let $\gamma : [-5\delta, 5\delta] \to \mathbb{R}^3$ be a smooth arc parametrized by arc length and let γ_δ be the restriction of γ to $[-\delta,\delta]$. Then*

$$c(\gamma_\delta) = \frac{1}{4\pi}\int\int_{u,v\in[-\delta,\delta]} \frac{|(\gamma'(v),\gamma'(u),\gamma(v)-\gamma(u))|}{|\gamma(v)-\gamma(u)|^3}dudv \le \frac{2}{3\pi}E(\gamma). \tag{31}$$

Sketch: First, by reparametrization we can assume that $\delta = 1/4$, so for any $x \in [-\delta,\delta] = [-1/4, 1/4]$ and $s \in [-1,1]$, we define an auxiliary function $G_x(s), G_x : [-1,1] \to \mathbb{R}^3$. This function gives us the energy relative to $\gamma(x)$ of the curve $\gamma_{x;x+s}$, which is defined as the segment of the knot restricted to the interval $[x, x+s]$ joined with a straight line segment of length $1-s$ in the direction $\gamma(x+s)-\gamma(x)$. The idea is that we then find an estimate for the integrand of equation 25 in terms of quantities such as $dG_x(s)/ds$, and this estimate will integrate to give an estimate for $c(\gamma)$ in terms of the energy. Although the auxilliary function appears somewhat unmotivated, the intuition is that it is related to the energy of γ and we can use it to get an estimate of the integrand of the crossing number equation, which will lead us to an estimate of the crossing number in terms of the energy. For more detail, the reader can consult Freedman, He, and Wang [6].

3.2 Minimization of Knot Energies

We have seen that there exists an energy-minimizing link; our next task is to show the existence of a similar energy minimizer for knots as well. Since knots are just links of one component, we might think that such an existence proof would follow directly from the same proof for links. However, the definition of the energy of a knot is different from that of a link, and we will prove a slightly stronger result for knots, namely that *for every knot class*, there exists an energy-minimizing knot. The theorem for link energy gave us a universal minimizer, which for knots can be proved to be the circle, but here we will prove the stronger result. The theorems and proofs in this section are adapted from those in Freedman, He, and Wang [6].

One of the first steps in proving the existence of such a minimizer is proving that there we can have a minimizing knot in a sequence of knots; i.e. a subsequence of knots converges to a knot with minimal energy. Formally, we have the following lemma.

Lemma 3.5 *Let $\{\gamma_i\}, \gamma_i : \mathbb{S}^1 \to \mathbb{R}^3$, be a sequence of knots with uniformly bounded energy, parameterized by arc length. If $\{\gamma_i(0)\}$ is a bounded sequence of points, then there is a subsequence $\{\gamma_{i_k}\}$ of $\{\gamma_i\}$ which converges locally uniformly to a knot γ with*

$$E(\gamma) \leq \liminf_{i_k \to \infty} E(\gamma_{i_k}). \tag{32}$$

Proof: Since the knots in the sequence $\{\gamma_i\}$ have uniformly bounded energy, the sequence $\{E(\gamma_i)\}$ is a bounded sequence in $\mathbb{R}$, hence we can use the Bolzano-Weierstrass Theorem (see, for example, Bartle [1] for an exact statement and proof) to conclude that it has a convergent subsequence $\left\{E(\gamma_{i_j})\right\}$. Therefore $\lim_{i_j \to \infty} E(\gamma_{i_j})$ exists.

Now we will consider the sequence of knots $\left\{\gamma_{i_j}\right\}$. By Lemma 1.12 above, we have that since the knots have finite energy, they are L-bi-Lipschitz, which implies that the family of knots is uniformly equicontinuous. In addition, since $\left\{\gamma_{i_j}(0)\right\}$ is a bounded sequence of points, and the knots are parametrized by arc length, the sequence $\left\{\gamma_{i_j}\right\}$ is uniformly bounded, and we satisfy the conditions for the Arzela-Ascoli Theorem (Theorem 1.15). We can then conclude by this theorem that the sequence $\{\gamma_{i_j}\}$ has a convergent subsequence, say $\left\{\gamma_{i_{j_k}}\right\}$, that converges uniformly to a knot γ.

Now we must show that this knot γ satisfies equation 32. Consider equation 1, the definition of knot energy. For brevity's sake, denote by $G_\gamma(u, v)$

the integrand of this equation. Since the subsequence $\left\{\gamma_{i_{j_k}}\right\}$ converges locally uniformly to γ, we have

$$G_\gamma(u,v) = \lim_{i_{j_k}\to\infty} G_{\gamma_{i_{j_k}}}(u,v). \tag{33}$$

From equation 1 we can see that $G_{\gamma_{i_{j_k}}}(u,v)$ is a non-negative function for all i,j,k, so we can apply Fatou's Lemma (Theorem 1.14) to conclude that

$$\begin{aligned} E(\gamma) &= \int\int G_\gamma(u,v)dudv \\ &= \int\int \lim_{i_{j_k}\to\infty} G_{\gamma_{i_{j_k}}}(u,v)dudv \\ &= \int\int \liminf_{i_{j_k}\to\infty} G_{\gamma_{i_{j_k}}}(u,v)dudv \\ &\leq \liminf_{i_{j_k}\to\infty} \int\int G_{\gamma_{i_{j_k}}}(u,v)dudv \\ &= \liminf_{i_{j_k}\to\infty} E(\gamma_{i_{j_k}}), \end{aligned}$$

hence γ is an energy minimizer, and the lemma is proved. □

Now, the question might be, why is this lemma not enough to conclude that an energy-minimizing knot exists? The answer is that we do have an energy-minimizing knot, just not necessarily one in the same knot class as the sequence of knots. Even though the knots in the sequence may be in the same knot class, it is possible that when taking the limit as we do in this lemma, a part of the knot could "pull tight", thus changing the knot class. So, we need more to prove our main theorem.

The following theorem is the first substantive theorem of this section, and its proof offers good insight into the types of arguments needed to prove interesting results about knot energies. It will also be needed to prove our main theorem later in this section. This proof is expanded from that given by Freedman, He, and Wang [6].

Theorem 3.6 *Let $\gamma : \mathbb{S}^1 \to \mathbb{R}^3$ be a knot. If $E(\gamma)$ is finite, then γ is tame.*

Proof: To begin, let us lay out the structure of the proof, since it is built on four major moves. All of the vague statements made here will be clarified in the body of the proof. First, it will be enough to show that γ is locally tame, i.e. for every point $w = \gamma(x)$ on the knot, there is an open, tame subarc of γ that contains the point. Second, we must find a subarc around

w such that an extension of it has small energy. Third, we must show that the average crossing number of this subarc is sufficiently small that in some projection there are no self-crossings. Fourth, once we have this projection, we know that simple arcs in the plane are tame, and we then must pull back the planar neighborhood using the projection.

(1) We will not prove here that it is sufficient to prove that γ is locally tame, but the result comes from Theorem 7 in [3], which gives us that locally tame arcs in a triangulated 3-manifold with boundary are tame. In our case, this 3-manifold is a bounded subset of $\mathbb{R}^3$ that contains γ, and therefore it is sufficient for us to prove that γ is locally tame.

(2) Let $w = \gamma(x)$ be given, and let $\gamma_{w,\delta}$ be the subarc of γ of length 2δ centered about w. We want to show that for all $\epsilon > 0$ there is a δ_0 such that for all $\delta < \delta_0, E(\gamma_{w,5\delta}) < \epsilon$. The idea is that since we are looking at subarcs, the region over which we integrate in equation 1 gets smaller, so the energy must be limited. For simplicity's sake, say we are originally integrating over the interval $[0, 1]$, corresponding to the unit circle. So, we proceed by contradiction. Suppose that for some $\epsilon > 0, E(\gamma_{w,5\delta}) \geq \epsilon$ for all $\delta > 0$. Given any $M > 0$, we will show that $E(\gamma) > M$, contradicting the fact that $E(\gamma)$ is finite. Now, since the subarc $\gamma_{w,5\delta}$ is of length 10δ, and the total length of γ is 1 (since it is parametrized by arc length), there are at most $\frac{1}{10\delta}$ subarcs, each with energy at least ϵ. Then the total energy contribution of the 5δ-subarcs is less than or equal to $\frac{\epsilon}{2\delta}$, and by a result in Freedman, He, and Wang [6] (Lemma 1.4) the sum of the energies of the subarcs of a knot is less than or equal to the total energy of the knot, so we have that $\frac{\epsilon}{10\delta} \leq E(\gamma)$. Now choose δ_0 such that $5\delta_0 < \frac{\epsilon}{2M}$, and we then have

$$E(\gamma) \geq \frac{\epsilon}{10\delta_0} > \frac{\epsilon \cdot 2M}{2 \cdot \epsilon} = M, \tag{34}$$

hence $E(\gamma) \geq M$, a contradiction. So there is a δ such that a subarc of length 5δ has energy less than ϵ.

(3) We recall that $c(\gamma)$ is the average crossing number of a knot γ. Now that we have the subarc $\gamma_{w,5\delta}$ and the restriction $\gamma_{w,\delta}$, we can apply Lemma 3.4 to conclude that

$$c(\gamma_{w,\delta}) \leq \frac{2}{3\pi} E(\gamma_{w,5\delta}) \leq \frac{2\epsilon}{3\pi}. \tag{35}$$

By Lemma 3.3, we have that

$$c(\gamma_{w,\delta}) = \frac{1}{4\pi} \int\int_{\theta \in \mathbb{S}^2} n(\gamma_{w,\delta}; \theta) dS, \tag{36}$$

and combining equations 35 and 36, we have

$$\frac{2\epsilon}{3\pi} \geq \frac{1}{4\pi} \int \int_{\theta \in \mathbb{S}^2} n(\gamma_{w,\delta}; \theta) dS, \tag{37}$$

which implies that

$$\frac{8}{3}\epsilon \geq \int \int_{\theta \in \mathbb{S}^2} n(\gamma_{w,\delta}; \theta) dS. \tag{38}$$

So, if we choose ϵ such that $\epsilon < \frac{3}{8}$, then $\int \int_{\theta \in \mathbb{S}^2} n(\gamma_{w,\delta}; \theta) dS < 1$, and therefore for some $\theta \in \mathbb{S}^2, n(\gamma_{w,\delta}; \theta) = 0$; i.e. there are no crossings when the subarc is projected in the θ-direction.

(4) Now by projecting the arc $\gamma_{w,\delta}$ in the θ-direction, we have a simple arc in $\mathbb{R}^2$, which is necessarily tame (see, e.g., Moise [13]). So we have an extension of the embedding $(-\delta, \delta) \to \mathbb{R}^2$ to an embedding $(-\delta, \delta) \times \mathbb{R} \to \mathbb{R}^2$. We can extend this product structure into $\mathbb{R}^3$ by crossing it with an interval $(-\alpha, \alpha), \alpha \in \mathbb{R}$, in the θ-direction, thus "thickening" the neighborhood about the projeted arc to a tubular neighborhood about the arc in $\mathbb{R}^3$. Thus the knot is locally tame, hence tame, and the theorem is proved. $\square$

We are almost ready to begin the proof of the main theorem of this section, the existence of an energy-minimizing knot for each knot class. We will give one more definition, followed by a lemma that will relate this definition to knots with finite energy, hence to tame knots and so to the rest of the results of this section.

Definition 3.7 *Let $B \subset \mathbb{S}^3 = \mathbb{R}^3 \cup \{\infty\}$ be a closed topological ball; i.e. B is homeomorphic to the unit ball $\mathbb{B}^3 = \{w \in \mathbb{R}^3 \mid |w| \leq 1\}$. A knot γ has its knot type captured in the topological ball B if $(\mathbb{S}^3 \setminus int(B); \gamma \setminus int(B))$ is homeomorphic to the unknotted pair $(\mathbb{B}^3; J)$, where $J \subset \mathbb{R}^3$ and $J = \{(w, 0, 0) \mid -1 \leq w \leq 1\}$.*

Now we have a lemma that relates the notion of having the knot type captured in a topological ball to knots with finite energy. We will give a sketch of the proof given by Freedman, He, and Wang [6].

Lemma 3.8 *For any $M > 0$ there is some $\delta > 0$ such that if γ is a knot with $E(\gamma) \leq M$, then there is a Mobius transformation T of $\mathbb{S}^3$ that takes γ to $\tilde{\gamma} \subset \mathbb{R}^3$ such that $\ell(\tilde{\gamma}) = 1$ and no closed ball of diameter $\leq \delta$ captures the knot type of $\tilde{\gamma}$.*

Sketch: We can prove (see Lemma 4.5 of Freedman, He, and Wang [6]) that for any tame knot there is a $\delta > 0$ such that no closed topological ball of spherical diameter $\leq \delta$ captures the knot type of the knot. In this lemma, we want to show that this δ is only dependent on the energy M of the knot, and we can transform any knot with energy less than M to a knot of length 1 with the above property. We can transform a knot γ to $\tilde{\gamma}$ such that the unit ball centered at 0 captures the knot type of the knot, but that no topological ball of radius 1/2 captures the knot type. Then for some δ_1 we construct a round sphere S disjoint from $\tilde{\gamma}$ such that the radius of the sphere is at least δ_1 and the distance from 0 to its center is at most 1. Then we use an inversion on this sphere and find a δ_2 dependent only on M such that the knot type of $\tilde{\gamma}$ under this inversion is not captured in any closed topological ball of diameter $\leq \delta_2$. We can then rescale in order to obtain the unit length condition, and the lemma is proved. The reader can consult Freedman, He, and Wang [6] for more details.

Before we move to the proof of the main theorem for knots, we will state a lemma that will be needed in its proof, although we will leave the proof of the lemma to the appendix, since it is of a group-theoretic nature and does not fit with the rest of this paper.

Lemma 3.9 *Let $\mathcal{N}$ be a regular tubular neighborhood of a knot γ; i.e. a neighborhood homeomorphic to a solid torus in which γ is unknotted, and let $\mathcal{M} \subset \mathcal{N}$ be homeomorphic to a solid torus containing γ. Then $\mathcal{M}$ is a regular tubular neighborhood of γ.*

Now we can prove the main theorem of this section. We restate it below for convenience. The proof given is adapted and expanded from that given by Freedman, He, and Wang [6].

Theorem 3.10 *Given a knot class K, there exists a knot $\gamma_K : \mathbb{S}^1 \to \mathbb{R}^3$ in K such that $E(\gamma_K) \leq E(\gamma)$ for any other knot γ in K.*

Proof: There are two main moves in this proof. First, we must show that there exists a loop with minimal energy; then, we must show that this minimal energy loop has the same knot type.

Let $\{\gamma_i\}$ be a sequence of loops in $\mathbb{R}^3$ parametrized by arc length of knot type K with energy approaching the infimum. The sequence $\{E(\gamma_i)\}$ is a decreasing sequence bounded below by 0, so we see that $\lim_{i\to\infty} E(\gamma_i)$ exists, and therefore there exists a uniform bound E on $E(\gamma_i)$. Then by Lemma 3.5, there exists a subsequence that converges uniformly to some closed curve γ_∞, with $E(\gamma_\infty) \leq \lim_{i\to\infty} E(\gamma_i)$, hence γ_∞ has minimal energy.

Now that we have a limit loop, we must show that it has the same knot type. So, we will show that γ_∞ is ambiently isotopic to γ_i for some

i. To do this, we show that γ_∞ and γ_i are both unknotted core curves for some neighborhood $\mathcal{M} \subset \mathbb{R}^3$, where $\mathcal{M}$ is homeomorphic to a solid torus. Since γ_∞ has finite energy, by Theorem 3.6 it is tame, and so there exists a regular tubular neighborhood $\mathcal{N}$ of γ_∞. Since the sequence of γ_i's converges uniformly to γ_∞, we can find an i_0 such that $\gamma_{i_0} \subset \mathcal{N}$. However, this neighborhood $\mathcal{N}$ may not be restrictive enough to guarantee that γ_{i_0} is also unknotted in $\mathcal{N}$, so we construct a locally straight, closed, regular tubular neighborhood $\mathcal{M}$ by joining together thin solid cylinders of length l and radius r (see figure 7 below).

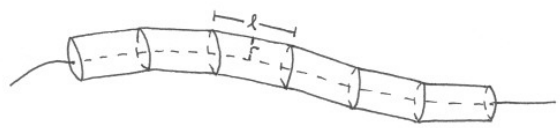

Figure 7: Arc inside joined solid cylinders

We consider points on γ_∞ separated by some small $l > 0$, where l is the distance in $\mathbb{R}^3$, and let the cylinders be of length l and have for axes the line segments between consecutive points. Since $\mathcal{N}$ is a regular tubular neighborhood of γ_∞ and $\mathcal{M} \subset \mathcal{N}$ is a solid torus containing γ_∞, by Lemma A.4 $\mathcal{M}$ is a regular tubular neighborhood of γ_∞, hence γ_∞ is an unknotted core curve for $\mathcal{M}$. When we construct this neighborhood below, we will choose r,l, and the number of cylinders in order to guarantee that γ_{i_0} is also an unknotted core curve of $\mathcal{M}$, hence γ_{i_0} is isotopic to γ_∞.

Now we will choose the appropriate regular tubular neighborhood $\mathcal{M}$ such that γ_{i_0} is an unknotted core curve of $\mathcal{M}$. To do this, we must determine the conditions for $\mathcal{M}$ under which this is true. First, we construct $\mathcal{M}$ as above as the union of adjacent closed cylinders C_n, where $n = 1, 2, \ldots, N$. By the earlier Lemma 3.8, there is some $\delta > 0$ such that no closed topological ball of diameter $\leq \delta$ captures the knot type of γ_{i_0}. So to ensure that γ_{i_0} is an unknotted core curve of $\mathcal{M}$, we will require that the diameter of $\mathcal{M}$ is less than δ, which we can accomplish by making the length l and the radius r of each cylinder sufficiently small. To do this, we first notice that for some fixed l and r, which determine the number of cylinders N, the diameter of $\mathcal{M}$ is at most the sum of the main diagonals of all the cylinders, so we have

$$d(\mathcal{M}) \leq 1000E \cdot \sqrt{l^2 + 4r^2}, \tag{39}$$

where $1000E \leq N$. We then can shrink $\mathcal{M}$ further by decreasing l and r, which will only decrease the diameter of $\mathcal{M}$ and increase N, but will not change the energy E. So, if we choose l, r such that $1000E \cdot \sqrt{l^2 + 4r^2} \leq \delta$, we have the appropriate condition. However, $\mathcal{M}$ is not itself a topological ball, so we must divide $\mathcal{M}$ into at least two topological balls using embedded disks that meet γ_{i_0} transversally in exactly one point. To do this, there must be only one spanning arc of γ_{i_0} in each cylinder C_n, where a spanning arc is an arc joining both faces of C_n. In figure 8 below, we see that γ_1 is a spanning arc, while γ_2 and γ_3 are not spanning arcs.

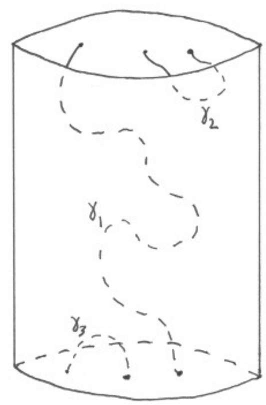

Figure 8: Cylinder with spanning arc

By Corollary 1.13 above, we have that since γ_{i_0} has finite energy, for any $\epsilon > 0$ there exists an η such that any subarc of γ_{i_0} of length η is a $(1 + \epsilon)$-bi-Lipschitz embedding. In our case, we let $\epsilon = 0.1$, so we can find $\eta > 0$ such that if $l < \eta$, then $\ell(\gamma_{i_0} \cap C_n) \leq 1.1l$. So, we choose l, r not only such that $1000E \cdot \sqrt{l^2 + 4r^2} \leq \delta$, but also such that $l < \eta$.

Now since $\ell(\gamma_{i_0} \cap C_n) \leq 1.1l$, we see that there can only be one spanning arc in C_n, and we can construct an embedded disk inside C_n that meets γ_{i_0} transversally in exactly one point. We will show that there is some θ with θ almost perpendicular to the axis of C_n such that the θ-projection of $\gamma_{i_0} \cap C_n$ has no self-crossings, and then we can construct an arc along the back wall of the projected cylinder that meets $\pi_\theta (\gamma_{i_0} \cap C_n)$ in exactly one point. Then we can pull back this arc using the inverse of the projection π_θ to construct an embedded disk that meets $\gamma_{i_0} \cap C_n$ transversally in exactly one point. Since the existence of this projection depends on the average crossing number of γ_{i_0}, we may not be able to get the angle of projection exactly perpendicular, but we will make the crossing number so small that it will be almost perpendicular. We want this projection to project the subarc $\gamma_{i_0} \cap C_n$ injectively into the cylinder wall ∂C_n, but since θ may not be exactly perpendicular to the axis of the cylinder, we consider instead the subarc $\gamma_{i_0} \cap \left\{ {D^2}_r \times [0.1l, 0.9l] \right\}$, which is the subarc of γ_{i_0} within the subcylinder of length $0.8l$. Since $\ell(\gamma_{i_0} \cap C_n) \leq 1.1l$, it follows that π_θ will project $\gamma_{i_0} \cap \left\{ {D^2}_r \times [0.1l, 0.9l] \right\}$ injectively into the cylinder wall ∂C_n.

By Lemma 3.3, in order for there to be a projection π_θ such that

$$\pi_\theta \left(\gamma_{i_0} \cap \left\{ {D^2}_r \times [0.1l, 0.9l] \right\} \right)$$

has no self crossings, the average crossing number of $\gamma_{i_0} \cap C_n$ must be sufficiently small, and by Lemma 3.4, the crossing number is dependent only on the energy of the curve. If $c\left(\gamma_{i_0} \cap C_n\right) \leq 0.005$ (for example), then such a projection exists. Now consider the subarc

$$\gamma_{i_0} \cap \left(\bigcup_{m=n-2}^{n+2} C_m \right).$$

By Lemma 3.4 we have

$$c\left(\gamma_{i_0} \cap C_n\right) \leq \frac{2}{3\pi} E \left(\gamma_{i_0} \cap \left(\bigcup_{m=n-2}^{n+2} C_m \right) \right), \tag{40}$$

and so if we have

$$E \left(\gamma_{i_0} \cap \left(\bigcup_{m=n-2}^{n+2} C_m \right) \right) \leq 0.01, \tag{41}$$

then

$$c\left(\gamma_{i_0} \cap C_n\right) \leq \frac{2}{3\pi} \cdot 0.01 \leq 0.005. \tag{42}$$

Since we have a uniform bound E on the energy of γ_{i_0}, we know that for the total number of cylinders N,

$$E \left(\gamma_{i_0} \cap \left(\bigcup_{m=1}^{N} C_m \right) \right) \leq E, \tag{43}$$

and therefore as long as $N \geq 500E + 2$, there are at least 2 cylinders that satisfy equation 41, say C_a and C_b.

Now that we have this projection π_θ, there are arcs β_a and β_b that span the θ-back walls of C_a and C_b, respectively, and meet the projected image of γ_{i_0} transversally in exactly one point. So $\beta_a{}' = \pi_\theta^{-1}(\beta_a)$ is an embedded disk separating C_a that meets γ_{i_0} transversally in exactly one point, and $\beta_b{}' = \pi_\theta^{-1}(\beta_b)$ is an embedded disk separating C_b with the same property.

To recap, then, we have at least two embedded disks that meet γ_{i_0} transversally in exactly one point and that separate $\mathcal{M}$ into topological balls of diameter less than δ. The curve γ_{i_0} meets each of these topological balls in an arc, which is unknotted because the knot type of γ_{i_0} is not captured in a closed topological ball of diameter less than δ. Therefore γ_{i_0} is an unknotted core curve of $\mathcal{M}$, and since γ_∞ is also an unknotted core curve of $\mathcal{M}$, γ_∞ and γ_{i_0} are ambiently isotopic, hence they have the same knot type, and the proof is complete. □

4 Comparison of Proof Structure

As a way of tying together the two main results of this paper, we will analyze and compare the structure of the proofs of these two results. We see right away that the proofs begin with a similar method; both start out with a sequence of knots or links with energy approaching the infimum, conclude using the Arzela-Ascoli Theorem that these sequences have convergent subsequences, and then apply Fatou's Lemma to show that the knot or link to which they converge is in fact an energy minimizer. The difference is that while this conclusion essentially ends the link proof, it only begins the knot proof. The main force of the link proof is in proving that this sequence of links satisfies the conditions of the Arzela-Ascoli Theorem, which we do by bounding the lengths of each element of the links. The only machinery that is necessary for this proof (besides the theorems mentioned above) is the notion of Mobius transformations. There are no difficult lemmas and no real heavy machinery needed.

We contrast the relative straightforwardness of the link proof with the difficulty of the knot proof. The main source of the difficulty stems from the fact that we are proving the existence of an energy minimizing knot for each knot class, and so we must show that once we find an energy minimizer using the Arzela-Ascoli theorem and Fatou's Lemma, this minimizer is in fact in the appropriate knot class. The thrust of the proof lies in constructing a neighborhood about the energy minimizing knot that guarantees that it is in the same knot class as the sequence. In order to accomplish this, the proof makes use of six different lemmas and theorems, none of which are particularly straightforward. We move between algebra, analysis, and topology throughout the proof, combining notions from all the disciplines to finally reach our conclusion.

A Group Theory

Since the group theory aspects of the proof of Theorem 3.10 do not exactly fit in with the rest of this paper, we leave them to an appendix. Here we will define free and amalgamated products, give a statement of the Seifert-van Kampen Theorem, and prove the group theory lemma from section 3.2. Most of the definitions and other statements in this appendix are synthesized from those in Hall [7] and Munkres [14]. This particular statement of the Seifert-van Kampen Theorem comes from Bredon [4].

First, let us introduce a few basic ideas in group theory. Given a set of groups $\{G_\alpha\}_{\alpha \in J}$, the sequence $[x_1, x_2, \ldots, x_n]$, with each x_i in some group G_α, is called a word in the groups G_α. We want to define two equivalance relations for these words:

(1) $[x_1, x_2, \ldots, x_{i-1}, x_i, x_{i+1}, \ldots, x_n]$ is equivalent to

$$[x_1, x_2, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n]$$

if x_i is the identity element of some group G_α.

(2) $[x_1, x_2, \ldots, x_{i-1}, x_i, x_{i+1}, \ldots, x_n]$ is equivalent to

$$[x_1, x_2, \ldots, x_{i-1}, {x_i}^*, x_{i+2}, \ldots, x_n]$$

if x_i and x_{i+1} are in the same group G_α and $x_i x_{i+1} = {x_i}^*$ in G_α.

Two words a and b are equivalent if there is a sequence $a_1 = a, a_2, \ldots, a_n = b$ such that a_i and a_{i+1} are elementary equivalent for $i = 1, 2, \ldots, n-1$, and a word $[x_1, x_2, \ldots, x_n]$ is reduced if no x_i is the identity element in its group, and x_i and x_{i+1} belong to different groups for $i = 1, 2, \ldots, n-1$. We can see that it is possible to shorten a word until no group G_α contains both x_i and x_{i+1}, and $x_i \neq 1$ for all i, and this minimal length word is called a reduced word. In addition, we can show (see Hall [7]) that there is in fact a unique reduced word in every equivalence class.

Now that we understand a bit of the background to group theory, we give two important definitions of groups as products, the free product and the amalgamated free product.

Definition A.1 *Let $\{G_\alpha\}_{\alpha \in J}$ be a set of groups. We define a product on equivalence classes of words by concatenation using the equivalence relations (1) and (2) by letting $[x][y] = [xy]$, and this product forms a group. We call this group the free product of the groups G_α, and we write*

$$G = \prod_{\alpha \in J}^{*} G_\alpha.$$

In order to define the amalgamated product, we consider a set of groups $\{G_\alpha\}_{\alpha\in J}$ such that each G_α contains a subgroup U_α with each U_α isomorphic to a group U. Now we define another equivalence relation for words in the groups G_α:

(3) $[x_1, x_2, \ldots, x_{i-1}, x_i, x_{i+1}, \ldots, x_n]$ is equivalent to

$$[x_1, x_2, \ldots, x_{i-1}, y_i, x_{i+1}, \ldots, x_n]$$

if $x_i \in U_\alpha \subset G_\alpha$ and if $y_i \in U_\beta \subset G_\beta$ is such that x_i and y_i correspond to the same element $u \in U$ under the isomorphisms from U_α and U_β to U.

Using this equivalence relation as well, we define the amalgamated free product.

Definition A.2 *Let $\{G_\alpha\}_{\alpha\in J}$ be a set of groups. We define a product on equivalence classes of words by concatenation using the equivalence relations (1), (2), and (3) by letting $[x][y] = [xy]$, and this product forms a group. We call this group the amalgamated free product of the groups G_α with the amalgamated subgroup U, and we write*

$$G = \prod_{\alpha\in J}^{*U} G_\alpha.$$

We say that a word x in the amalgamated product is in canonical form if it is of the form $x = uz_1z_2\cdots z_n$, where $u \in U$ and each z_i is a coset representative in G_i. By a theorem of Hall [7] (Theorem 17.2.1), we have that for each class of equivalent words there is a unique element in canonical form.

Now that we have the background and definitions, we can give a statement of the Seifert-van Kampen Theorem, which can help us determine the fundamental group of a space X that can be written as the union of two open subsets with path-connected intersection. This particular statement comes from Bredon [4].

Theorem A.3 (Seifert-van Kampen Theorem) *Let $X = U \cup V$, where U, V and $U \cap V$ are all open, non-empty, and path-connected. Then the canonical inclusion maps of the fundamental groups of U, V, and $U \cap V$ into the fundamental group of X induce an isomorphism $\Psi : \pi_1(U) *_{\pi_1(U\cap V)} \pi_1(V) \to \pi_1(X)$.*

Now we will give the proof of the earlier lemma of section 3.2. We restate the lemma here.

Lemma A.4 *Let $\mathcal{N}$ be a regular tubular neighborhood of a knot γ, and let $\mathcal{M} \subset \mathcal{N}$ be a solid torus containing γ. Then $\mathcal{M}$ is a regular tubular neighborhood of γ.*

Proof: We will show that $\mathcal{M}$ is a regular tubular neighborhood of γ by showing that $\mathcal{M} \setminus \gamma$ is homotopic to a torus; that is, the fundamental group $\pi_1(\mathcal{M} \setminus \gamma)$ is isomorphic to $\mathbb{Z} \oplus \mathbb{Z}$. This proof will make use of the Seifert-van Kampen Theorem as stated above (Theorem A.3). To use this theorem, we need to define a set $X = U \cup V$, where U, V, and $U \cap V$ are open, non-empty, path-connected sets. We let $X = \mathcal{N} \setminus \gamma = (\mathcal{M} \setminus \gamma) \cup (\mathcal{N} \setminus \mathcal{M})$. Since $\mathcal{M}$ is open and γ is closed, the set $\mathcal{M} \setminus \gamma$ is an open, non-empty, path-connected set, so we let $U = \mathcal{M} \setminus \gamma$. However, the set $\mathcal{N} \setminus \mathcal{M}$ is not open, so we take a deformation retract $\mathcal{M}'$ of $\mathcal{M}$ such that $\mathcal{M}' \subset \mathcal{M}$, and consider the closure $\overline{\mathcal{M}'}$. Then $\mathcal{N} \setminus \overline{\mathcal{M}'}$ is an open, non-empty, path-connected set, so we let $V = \mathcal{N} \setminus \overline{\mathcal{M}'}$. So, up to homotopy, we have $X = \mathcal{N} \setminus \gamma = (\mathcal{M} \setminus \gamma) \cup (\mathcal{N} \setminus \overline{\mathcal{M}'})$, and $U \cap V = (\mathcal{M} \setminus \gamma) \cap (\mathcal{N} \setminus \overline{\mathcal{M}'}) = \partial \mathcal{M}$. Therefore, by the Seifert-van Kampen Theorem, there exists an isomorphism $\Psi : \pi_1(\mathcal{M} \setminus \gamma) *_{\pi_1(\partial \mathcal{M})} \pi_1(\mathcal{N} \setminus \overline{\mathcal{M}'}) \to \pi_1(\mathcal{N} \setminus \gamma)$.

Now, since $\mathcal{N}$ is a regular tubular neighborhood of γ, $\mathcal{N} \setminus \gamma$ is homotopic to a torus, so $\pi_1(X) = \pi_1(\mathcal{N} \setminus \gamma) \cong \mathbb{Z} \oplus \mathbb{Z}$. Again, since $\partial \mathcal{M}$ is homotopic to a torus, $\pi_1(U \cap V) = \pi_1((\mathcal{M} \setminus \gamma) \cap (\mathcal{N} \setminus \overline{\mathcal{M}'})) = \pi_1(\partial \mathcal{M}) \cong \mathbb{Z} \oplus \mathbb{Z}$. So, in fact we have that the amalgamated product of $\pi_1(U)$ and $\pi_1(V)$ amalgamated over $\mathbb{Z} \oplus \mathbb{Z}$ is isomorphic to $\mathbb{Z} \oplus \mathbb{Z}$.

Consider the following commutative diagram, given by the Seifert-van Kampen Theorem:

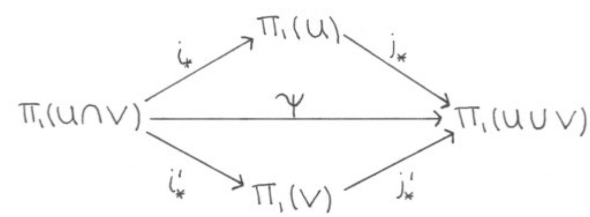

Figure 9: Seifert-van Kampen diagram

We represent elements in $\pi_1(U)$ and $\pi_1(V)$ by their cosets with respect to $i_*(\pi_1(U \cap V))$ and $i'_*(\pi_1(U \cap V))$, respectively, and we represent elements in $\pi_1(U \cup V)$ by two cosets. So, we choose coset representatives $s \in \pi_1(U)$ and $s' \in \pi_1(V)$, and since $\pi_1(U \cap V)$ is a subgroup of $\pi_1(U)$ and $\pi_1(V)$, we represent elements in $\pi_1(U)$ by us and elements in $\pi_1(V)$ by us', where $u \in \pi_1(U \cap V)$. By Theorem 17.2.1 from Hall [7], which gives us the uniqueness of words in canonical form, we can represent elements in $\pi_1(U \cup V)$ by uss'. Now, consider the element $us' \in \pi_1(V)$. Then since the inclusion map $j'_* : \pi_1(V) \to \pi_1(U \cup V)$ (see figure 9 above) is a homomorphism, it must map the identity element in $\pi_1(V)$ to the identity element in $\pi_1(U \cup V)$, so we have $j'_*(us') = u1s'$, where 1 is the identity element in $\pi_1(U)$. Since the map $\Psi = j'_* \circ i'_*$ is an isomorphism, j'_* is onto, hence all elements of $\pi_1(U \cup V)$ must be of the form $u1s'$. Therefore, by the uniqueness of words in canonical form, any coset representative in $\pi_1(U)$ must equal 1, so there is only one coset in $\pi_1(U)$, and therefore $\pi_1(U)$ is isomorphic to $\pi_1(U \cap V) \cong \mathbb{Z} \oplus \mathbb{Z}$. By a theorem from Hempel [9] (Theorem 10.2), we have that since $\pi_1(\mathcal{M} \setminus \gamma) = \pi_1(U) \cong \mathbb{Z} \oplus \mathbb{Z}$, the pair $(\mathcal{M}, \gamma)$ is homeomorphic to the standard pair $(\mathbb{S}^1 \times \mathbb{R}^2, \mathbb{S}^1 \times \{0\})$, which is the standard solid torus and its standard core curve. Therefore, we can conclude that $\mathcal{M}$ is a regular tubular neighborhood of γ, and the lemma is proved. $\square$

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